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ON POLYNOMIAL AUTOMORPHISMS OF SPHERES

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ABSTRACT

Let K be an infinite field with characteristic different from two and $\mathbb{S}^n(K)$ the n-sphere over K. We show that ambient polynomial automorphisms of $\mathbb{S}^n(K)$ preserve the quadratic form $x_0^2+\cdots+x_n^2$ and the group $\mathrm{Aut}\,(K^{n+1},\mathbb{S}^n(K))$ of such automorphisms of $\mathbb{S}^n(K)$ is isomorphic to the $(n+1)$-orthogonal group $O(n+1,K)$ provided K is real.

Next, the restriction map $\mathrm{Aut}(K^3,\mathbb{S}^2(K)) \to \mathrm{Aut}\,(\mathbb{S}^2(K))$ yields a surjection provided K is an algebraically closed field as well. Furthermore, for any such a field K, there is an imbedding

$$K[X_1\dots,X_m]^{\frac{m(m-1)}{2}+mn} \longrightarrow \mathrm{Aut}\,(K^{2m+n},\mathbb{S}^{2m+n-1}(K)).$$

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Introduction

Given an algebra A over a field K, a description of the automorphism group $\mathrm{Aut}\,(A)$, one of its most important characteristic, is really a very hard problem. A geometric counterpart of that is a description of polynomial automorphisms $\mathrm{Aut}\,(V)$ of the corresponding affine variety V. Though that problem for the plane K^2 was settled in [4, 5], the problem of finding all polynomial automorphisms of K^3 is still open. By [8], polynomial automorphisms of hypersurface of degree $d \geq 3$ in the $(n+1)$-projective space are finite except for some cases. In the light of [1], the automorphism group $\mathrm{Aut}\,(\mathbb{S}^1(K))$ of the circle $\mathbb{S}^1(K)$ over an infinite field K is isomorphic to the orthogonal group $O(2,K)$. Furthermore, by the results of [2], the circle is the only compact connected curve with such an infinite group provided K is the field of reals.

Polynomial automorphisms of the hypersurface in K^3 given by $x_0^n x_1 = P(x_2)$ with $n \geq 1$ were considered first in [6] for $n = 1$ and then in [7] for $n > 1$. In particular, automorphisms of the 2-dimensional sphere $\mathbb{S}^2(K)$ over K could be derived, provided $i \in K$ with $i^2 = -1$. In the light of [2], given a hypersurface V determined by a proper polynomial map $\mathbb{R}^{n+1} \to \mathbb{R}$ over the reals $\mathbb{R}$ for $n \geq 0$, the group $\mathrm{Aut}\,(\mathbb{R}^{n+1}, V)$ of ambient automorphism of V is isomorphic to an algebraic subgroup of the orthogonal group $O(q)$ for some q. The aim of this note is studying the automorphism groups $\mathrm{Aut}\,(\mathbb{S}^n(K))$ and ambient automorphisms $\mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K))$, where $\mathbb{S}^n(K)$ is the n-sphere over a field K.

Section 1 investigates the group $\mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K))$ and studies the restriction map $\rho_n(K^{n+1}, \mathbb{S}^n(K)) : \mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K)) \to \mathrm{Aut}\,(\mathbb{S}^n(K))$. We generalize the result of [2] and derive in Proposition 1.5 that the group $\mathrm{Aut}(K^{n+1}, \mathbb{S}^n)$ coincides with polynomial automorphisms of K^{n+1} preserving the form $x_0^2 + \cdots + x_n^2$ provided K is an infinite field with characteristic different from two. We derive in Corollary 1.6 that $\mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K)) = O(n+1, K)$, provided K is a real field. In Proposition 1.10 we make use of [1] to show that

$$\rho_1(K^2, \mathbb{S}^1(K)) : \mathrm{Aut}\,(K^2, \mathbb{S}^1(K)) \to \mathrm{Aut}\,(\mathbb{S}^1(K))$$

is an isomorphism. Next, in Corollary 1.12 we derive from [6] that

$$\rho_2(K^3, \mathbb{S}^2(K)) : \mathrm{Aut}\,(K^3, \mathbb{S}^2(K)) \to \mathrm{Aut}\,(\mathbb{S}^2(K))$$

is a surjection for any algebraically closed field K with characteristic different from two.

Section 2 is devoted to polynomial automorphisms for higher dimensional spheres. In particular, an injective map

$$K[X_1,\dots,X_m]^{\frac{m(m-1)}{2}+mn} \longrightarrow \mathrm{Aut}\,(K^{2m+n},\mathbb{S}^{2m+n-1}(K))$$

of the $(\frac{m(m-1)}{2}+mn)$-th Cartesian power of the polynomial ring $K[X_1,\dots,X_m]$ endowed with an appropriate group structure is constructed.

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1. Ambient polynomial automorphisms

Let $K[X_0,\dots,X_n]$ be the polynomial ring in indeterminates $X_0,\dots,X_n$ over a field K. Given an affine irreducible variety $V \subseteq K^{n+1}$, write $\mathrm{Aut}\,(V)$ (resp., $\mathrm{Aut}\,(K^{n+1},V)$) for the group of polynomial (resp., ambient) automorphisms of V. Clearly, we have the restriction map

$$\rho_n(K^{n+1},V) : \mathrm{Aut}\,(K^{n+1},V) \to \mathrm{Aut}\,(V).$$

Note that the group $\mathrm{Aut}\,(V)$ is anti-isomorphic to the group of K-automorphisms of the ring $K[V]$ of regular functions on V provided K is algebraically closed.

Now, let

$$\mathbb{S}^n(K) = \{(x_0,\dots,x_n) \in K^{n+1};\, x_0^2+\cdots+x_n^2 = 1\}$$

be the n-**sphere** over K. If the field K is finite then it is well-known that $\mathrm{Aut}\,(\mathbb{S}^n(K))$ coincides with the group of self-bijections of $\mathbb{S}^n(K)$. For K with characteristic two, the sphere $\mathbb{S}^n(K)$ is actually the hyperplane given by $x_0+\cdots+x_n = 1$. Then, the polynomial map $\Phi : \mathbb{S}^n(K) \to K^n$ given by $\Phi(x_0,\dots,x_n) = (x_1,\dots,x_n)$ for $(x_0,\dots,x_n) \in \mathbb{S}^n(K)$ yields isomorphisms of the groups $\mathrm{Aut}\,(\mathbb{S}^n(K))$ and $\mathrm{Aut}\,(K^{n+1},\mathbb{S}^n(K))$ with $\mathrm{Aut}\,(K^n)$ and $\mathrm{Aut}\,(K^{n+1},K^n)$, respectively. Furthermore, the map Φ and the restriction map $\rho_n(K^{n+1},\mathbb{S}^n(K)) : \mathrm{Aut}\,(K^{n+1},\mathbb{S}^n(K)) \to \mathrm{Aut}\,(\mathbb{S}^n(K))$ lead to a splitting short exact sequence

$$1 \to \mathrm{Ker}\,\eta_n \longrightarrow \mathrm{Aut}\,(K^{n+1},K^n) \xrightarrow{\eta_n} \mathrm{Aut}\,(K^n) \to 1$$

of groups, where

$$(0,\eta_n(\varphi)(x_1,\dots,x_n)) = \varphi(0,x_1,\dots,x_n)$$

for any $\varphi \in \operatorname{Aut}(K^{n+1}, K^n)$ and $(x_1, \ldots, x_n) \in K^n$. Note that any polynomial $p_1(X_0), \ldots, p_n(X_0, \ldots, X_{n-1}) \in K[X_0, \ldots, X_n]$ gives rise to the polynomial automorphism

$$(\alpha X_0, X_1 + X_0 p_1(X_0), \ldots, X_n + X_0 p_n(X_0, \ldots, X_{n-1}))$$

if $\alpha \neq 0$. Such automorphisms form a subgroup of $\in \operatorname{Ker} \eta_n$.

Remark 1.1: If K is a finite field, then the restriction map

$$\rho_n(K^{n+1}, \mathbb{S}^n(K)) : \operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K)) \to \operatorname{Aut}(\mathbb{S}^n(K))$$

is obviously surjective but never injective.

For the rest of this paper we always assume the field K to be infinite and with characteristic different from two.

Let $\langle x, y\rangle = \sum_{i=0}^n x_i y_i$ for $x = (x_0, \ldots, x_n), y = (y_0, \ldots, y_n) \in K^{n+1}$. Write $G_{n+1}(K)$ for the monoid of all polynomial maps $\varphi : K^{n+1} \to K^{n+1}$ such that $\big\langle \varphi(x), \varphi(x)\big\rangle = \big\langle x, x\big\rangle$ for any $x \in K^{n+1}$ and observe that if K is a real field, then $\deg(\varphi) < 2$ for $\varphi \in G_{n+1}(K)$. We recall that a field K is called **real** if $-1 \neq x_1^2 + \cdots + x_n^2$ for any $x_1, \ldots, x_n \in K$. It is also clear that $O(n+1, K) \subseteq G_{n+1}(K)$, where $O(n+1, K)$ denotes the $(n+1)$-orthogonal group over K.

PROPOSITION 1.2: *If $\varphi \in \operatorname{Aut}(\mathbb{S}^n(K))$ is given by polynomials of global degree at most one, then $\varphi \in O(n+1, K)$.*

Proof. Let $\varphi = (\varphi_0, \ldots, \varphi_n)$ and write $\varphi_s(X_0, \ldots, X_n) = \sum_{t=0}^n a_{st} X_t + a_s$ for $s = 0, \ldots, n$ or in the matrix form $\varphi(X) = AX + a$, where $A = [a_{st}]_{0 \le s,t \le n}$, $a = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$ and $X = \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix}$. If $x \in \mathbb{S}^n(K)$, then certainly $\varphi(\pm x) \in \mathbb{S}^n(K)$. Hence, $1 = \langle \varphi(\pm x), \varphi(\pm x)\rangle = \langle \pm Ax + a, \pm Ax + a\rangle = \langle Ax, Ax\rangle \pm 2\langle Ax, a\rangle + \langle a, a\rangle$. This implies $\langle Ax, a\rangle = 0$ and $\langle Ax, Ax\rangle + \langle a, a\rangle = 1$ for all $x \in \mathbb{S}^n(K)$.

Now, we show that $\langle a, a\rangle = 0$. In fact, suppose that $\langle a, a\rangle \neq 0$ and consider $\frac{a}{\sqrt{\langle a,a\rangle}} \in \mathbb{S}^n(\overline{K})$, where $\overline{K}$ is the algebraic closure of K. By means of [1], the sphere $\mathbb{S}^n(K)$ is Zariski dense in $\mathbb{S}^n(\overline{K})$ so we may regard φ as an automorphism of $\mathbb{S}^n(\overline{K})$. But there is $x \in \mathbb{S}^n(\overline{K})$ with $\varphi(x) = Ax + a = \frac{a}{\sqrt{\langle a,a\rangle}}$ and so

$$\sqrt{\langle a, a\rangle} = \langle \varphi(x), a\rangle = \langle Ax, a\rangle + \langle a, a\rangle = \langle a, a\rangle.$$

Then $\langle a, a\rangle = 1$ and $\varphi(x) = Ax + a = a$ imply $Ax = 0$. Consequently, $\varphi(-x) = -Ax + a = a = \varphi(x)$. But φ is injective, so $-x = x$ and this leads to a contradiction because $x \in \mathbb{S}^n(\overline{K})$.

Therefore $A^tA = I_{n+1}$ i.e., $A \in O(n+1, K)$. This also implies $\langle x, A^t a\rangle = 0$ for all x, and so $a = 0$ and consequently $\varphi \in O(n+1, K)$. ∎

If $\varphi = (\varphi_0, \ldots, \varphi_n) : K^{n+1} \to K^{n+1}$ is a polynomial map, write

$$J\varphi = \begin{pmatrix} \frac{\partial \varphi_0}{\partial x_0} \cdots \frac{\partial \varphi_0}{\partial x_n} \\ \cdots\cdots\cdots \\ \frac{\partial \varphi_n}{\partial x_0} \cdots \frac{\partial \varphi_n}{\partial x_n} \end{pmatrix}$$

for its Jacobian.

Proposition 1.3: *If $\varphi \in G_{n+1}(K)$ then the following four conditions are equivalent:*

(a) $\varphi(0) = 0$;
(b) $\varphi^{-1}(0) \neq \emptyset$;
(c) $(J\varphi)(0) \in O(n+1, K)$;
(d) $\det(J\varphi)(0) \neq 0$, *where $J\varphi$ is the Jacobian of φ.*

Proof. The condition $\big\langle \varphi(x), \varphi(x)\big\rangle = \big\langle x, x\big\rangle$ for $x \in K^{n+1}$ implies

$$\frac{\partial}{\partial x_j}\langle \varphi(x), \varphi(x)\rangle = \frac{\partial}{\partial x_j}\langle x, x\rangle$$

or equivalently $\sum_{i=0}^{n} \varphi_i(x) \frac{\partial \varphi_i}{\partial x_j}(x) = x_j$ for $\varphi = (\varphi_0, \ldots, \varphi_n)$, $j = 0, \ldots, n$ and $x = (x_0, \ldots, x_n) \in K^{n+1}$. Therefore, we have in the matrix form

$$(1) \qquad (J\varphi)(x)^t \varphi(x) = x$$

for $x \in K^{n+1}$ and this clearly gives the equivalence of (a) and (b).

Then, we derive from (1)

$$\big\langle \varphi(x), (J\varphi)(x)x' \big\rangle = \big\langle x, x' \big\rangle$$

for any $x, x' \in K^{n+1}$ and so, replacing $x, x' \mapsto \lambda x$ we obtain

$$\lambda\langle \varphi(\lambda x), (J\varphi)(\lambda x)x\rangle = \lambda^2\langle x, x\rangle$$

for any $\lambda \in K$. Hence,

$$(2) \qquad \langle \varphi(\lambda x), (J\varphi)(\lambda x)x\rangle = \lambda\langle x, x\rangle$$

for all $\lambda \in K$ because both sides are polynomials in λ and the field K is infinite. Applying $\frac{d}{d\lambda}$ to (2), we obtain

$$\langle \frac{d}{d\lambda}(J\varphi)(\lambda x)x, \varphi(\lambda x)\rangle + \langle (J\varphi)(\lambda x)x, (J\varphi)(\lambda x)x\rangle = \langle x, x\rangle.$$

Now, for $\lambda = 0$, we have

$$\langle \frac{d}{d\lambda}(J\varphi)(\lambda x)x|_{\lambda=0}, \varphi(0)\rangle + \langle (J\varphi)(0)x, (J\varphi)(0)x\rangle = \langle x, x\rangle. \tag{3}$$

Then, from (3) we deduce that (a)$\Longrightarrow$(c).

The implication (c)$\Longrightarrow$(d) is obvious since $\det(J\varphi)(0) = \pm 1$ if $(J\varphi)(0) \in O(n+1, K)$.

To show (d)$\Longrightarrow$(a) observe that for $\lambda = 0$, (2) gives $\langle \varphi(0), (J\varphi)(0)x\rangle = 0$ for any $x \in K^{n+1}$; and since the matrix $(J\varphi)(0)$ is invertible, we obtain $\varphi(0) = 0$ and the proof is complete. ∎

Corollary 1.4: *Any element $\varphi \in G_{n+1}(K)$ such that $\varphi(0) = 0$ can be written as a product of an element in $O(n+1, K)$ and $\psi \in G_{n+1}(K)$ such that $\psi(0) = 0$ and $(J\psi)(0) = I_{n+1}$ and so $\psi(x) = x + \psi_2(x) + \cdots + \psi_t(x)$, where $\psi_d(x)$ is the homogeneous component of ψ with degree d for $d = 2, \ldots, t$.*

Furthermore, we may state:

Proposition 1.5: $\operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K)) = \operatorname{Aut}(K^{n+1}) \cap G_{n+1}(K)$.

Proof. The inclusion $\operatorname{Aut}(K^{n+1}) \cap G_{n+1}(K) \subseteq \operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K))$ is clear. To show the opposite inclusion, given $\varphi = (\varphi_0, \ldots, \varphi_n) \in \operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K))$, the polynomial $\sum_{i=0}^n \varphi_i^2 - 1$ is irreducible because $\sum_{i=0}^n X_i^2 - 1$ is so. But $\sum_{i=0}^n \varphi_i^2 - 1$ vanishes on the sphere $\mathbb{S}^n$ and so

$$\sum_{i=0}^n \varphi_i^2 - 1 = \alpha\Big(\sum_{i=0}^n X_i^2 - 1\Big)$$

for some $\alpha \in K^*$. Then we derive, as in the proof of Proposition 1.3:

$$(J\varphi)(x)^t \varphi(x) = \alpha x \tag{4}$$

for all $x \in K^{n+1}$. If we choose now $x \in K^{n+1}$ such that $\varphi(x) = 0$, (4) gives $x = 0$. Therefore, $\varphi(0) = 0$ and so $\alpha = 1$ and then $\varphi \in G_{n+1}(K)$. ∎

Then, by Proposition 1.5 and the obvious degree argument consideration, we can derive:

Corollary 1.6: *If K is a real field, then $G_{n+1}(K) = O(n+1, K) = \mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K))$.*

The following example shows that if $i \in K$ then there are polynomial maps $\varphi \in G_{n+1}(K)$ with $\varphi(0) \neq 0$ and so $G_{n+1}(K) \nsubseteq \mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K))$ in general.

Example 1.7: For $n \geq 2$, consider the polynomial map $\varphi : K^{n+1} \longrightarrow K^{n+1}$ given by

$$\varphi(x_0, \ldots, x_n) = \Big(\frac{1}{2}(x_0^2 + \cdots + x_n^2\Big), i\Big(\frac{1}{2}(x_0^2 + \cdots + x_n^2) - 1\Big), 1, 0, \ldots, 0)$$

for any $(x_0, \ldots, x_n) \in K^{n+1}$.

It is obvious that $\varphi \in G_{n+1}(K)$ but $\varphi(0) = (0, -i, 1, 0, \ldots, 0)$.

Given a real field K, write $K(i)$ for its extension with $i^2 = -1$, $U(n+1, K(i))$ for the $(n+1)$-st unitary group over the field $K(i)$ and $U(K(i)^{n+1}, K \times K(i)^n)$ for the subgroup of $U(n+1, K(i))$ formed by ambient maps with respect to $K \times K(i)^n$. If $\varphi = (\varphi_0, \ldots, \varphi_n) \in \mathrm{Aut}\, K(i)^{n+1}$ and

$$\begin{aligned}\varphi_j &= \varphi_j(X_0 + iX_0', \ldots, X_n + iX_n') \\ &= \varphi_j'(X_0, \ldots, X_n, X_0', \ldots, X_n') + i\varphi_j''(X_0, \ldots, X_n, X_0', \ldots, X_n')\end{aligned}$$

for $j = 0, \ldots, n$, then we get $(\varphi_1', \varphi_1'', \ldots, \varphi_n', \varphi_n'') \in \mathrm{Aut}\,(K^{2n+2})$. Because $\mathbb{S}^{2n+1}(K) \subseteq K(i)^{n+1}$ and $\mathbb{S}^{2n}(K) \subseteq K \times K(i)^n$, in view of Corollary 1.6, we can state:

Remark 1.8: Let K be a real field. Then, $\mathrm{Aut}\,(K(i)^{n+1}, \mathbb{S}^{2n+1}(K)) = U(n+1, K(i))$ and $\mathrm{Aut}\,(K \times K(i)^n, \mathbb{S}^{2n}(K)) = U(K(i)^{n+1}, K \times K(i)^n)$.

By [3], an affine variety $V \subseteq K^{n+1}$ is called an **identity set** for polynomial automorphisms of K^{n+1} if the restriction map

$$\rho_n(K^{n+1}, V) : \mathrm{Aut}\,(K^{n+1}, V) \to \mathrm{Aut}\,(V)$$

is a monomorphism. Thus, in the light of the discussion above, we can state:

Theorem 1.9: *Let K be a real field. Then, for $n \geq 0$:*

(a) *$\mathbb{S}^n(K)$ is an identity set for polynomial automorphisms of K^{n+1};*
(b) *$\mathbb{S}^{2n+1}(K)$ is an identity set for polynomial automorphisms of $K(i)^n$.*
(c) *$\mathbb{S}^{2n}(K)$ is an identity set for polynomial automorphisms of $K \times K(i)^n$.*

Furthermore, it holds:

Proposition 1.10: *The restriction map*

$$\rho_1(K^2,\mathbb{S}^1(K)):\mathrm{Aut}(K^2,\mathbb{S}^1(K))\to \mathrm{Aut}\,(\mathbb{S}^1(K))$$

is an isomorphism.

Proof. In virtue of [1], it holds $\mathrm{Aut}\,(\mathbb{S}^1(K)) = O(2,K)$. Hence, the map $\rho_1(K^2,\mathbb{S}^1(K)):\mathrm{Aut}\,(K^2,\mathbb{S}^1(K))\to \mathrm{Aut}\,(\mathbb{S}^1(K))$ is surjective because $O(2,K)\subseteq \mathrm{Aut}\,(K^2,\mathbb{S}^1(K))$.

Now, if $\varphi=(\varphi_0,\varphi_1)\in \mathrm{Ker}\,\rho_1(K^2,\mathbb{S}^1(K))$, then, by Proposition 1.5, $\varphi_0^2+\varphi_1^2 = X_0^2+X_1^2$. Because $\mathbb{S}^1(K)$ is Zariski dense in $\mathbb{S}^1(\overline{K})$, we may assume that $i\in K$ with $i^2=-1$. Hence, $(\varphi_0+i\varphi_1)(\varphi_0-i\varphi_1) = X_0^2+X_1^2$. But φ_0 and φ_1 are algebraically independent and so $\deg\,(\varphi_0+i\varphi_1)=\deg\,(\varphi_0-i\varphi_1)=1$ and $\deg\varphi_0=\deg\varphi_1=1$. By Proposition 1.2, we get that $\varphi\in O(2,K)$. But $\varphi\in \mathrm{Ker}\rho_1(K^2,\mathbb{S}^1(K))$ so φ is the identity automorphism of K^2. ∎

Now, we aim to show the surjectivity of the restriction map

$$\rho_2(K^3,\mathbb{S}^2(K)):\mathrm{Aut}\,(K^3,\mathbb{S}^2(K))\to \mathrm{Aut}\,(\mathbb{S}^2(K))$$

for any algebraically closed field K. We point out that over such a field the 2-sphere $\mathbb{S}^2(K)$ might be described by $x_0x_1+x_2^2=1$.

Given an algebraically closed field K, Makar-Limanov [6] considers the factor K-algebra $K[X_0,X_1,X_2]/(X_0X_1-p(X_2))$ for any polynomial $p\in K[X]$. If $p(X_2)=1-X_2^2$ then [6, Theorem] yields:

Corollary 1.11: *Let K be an algebraically closed field K.*

Then, the group $\mathrm{Aut}(K[X_0,X_1,X_2]/(X_0X_1+X_2^2-1))$ *is generated by the following automorphisms:*

1. *hyperbolic rotations* $H_\lambda(X_0)=\lambda X_0$, $H_\lambda(X_1)=\lambda^{-1}X_1$, $H_\lambda(X_2)=X_2$ *for* $\lambda\in K^*$;
2. *involution* $I(X_0)=X_1$, $I(X_1)=X_0$, $I(X_2)=X_2$;
3. *the symmetry* $S(X_0)=X_0$, $S(X_1)=X_1$, $S(X_2)=-X_2$;
4. *triangular* $\Delta_p(X_0)=X_0$, $\Delta_p(X_1)=X_1-2X_2p(X_0)-X_0p^2(X_0)$, $\Delta_p(X_2)=X_2+X_0p(X_0)$ *for* $p(X)\in K[X]$.

Now, consider the isomorphism of K-algebras

$$K[X_0,X_1,X_2]/(X_0X_1+X_2^2-1)\xrightarrow{\cong} K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1)$$

given by the assignment $(X_0, X_1, X_2) \mapsto (X_0 + iX_1, X_0 - iX_1, X_2)$. Then, its inverse sends $(X_0, X_1, X_2) \mapsto (\frac{X_0+X_1}{2}, \frac{X_0-X_1}{2i}, X_2)$.

Thus, those four types of the maps above generate

$$\mathrm{Aut}\,(K[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2 - 1))$$

and they certainly lead to generators of the group $\mathrm{Aut}\,(\mathbb{S}^2(K))$ as well. But any of those generators is extensible to a polynomial automorphism of K^3. Thus, in view of Corollary 1, we can state:

Corollary 1.12: *Let K be an algebraically closed field K. Then, the restriction map*

$$\rho_2(K^3, \mathbb{S}^2(K)) : \mathrm{Aut}\,(K^3, \mathbb{S}^2(K)) \to \mathrm{Aut}\,\mathbb{S}^2(K)$$

is a surjection.

By [1], the sphere $\mathbb{S}^2(K)$ is Zariski dense in $\mathbb{S}^2(\overline{K})$, where $\overline{K}$ is the algebraic closure of K. Thus, any polynomial automorphism $\varphi \in \mathrm{Aut}\,(\mathbb{S}^2(K))$ yields $\varphi' \in \mathrm{Aut}\,(\mathbb{S}^2(\overline{K}))$ rising, in the light of the above, a polynomial automorphism of $\overline{K}^3$ as well. But we cannot say that φ' restricts to a polynomial automorphism of K^3 to claim that the restriction map $\rho_2(K^3, \mathbb{S}^2(K)) : \mathrm{Aut}\,(K^3, \mathbb{S}^2(K)) \to \mathrm{Aut}\,(\mathbb{S}^2(K))$ is a surjection.

For a real field K, by Corollary 1.6, $\mathrm{Aut}(K^{n+1}, \mathbb{S}^n(K)) = O(n+1, K)$. Hence, $\rho_n(K^{n+1}, \mathbb{S}^n(K)) : \mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n(K)) \to \mathrm{Aut}\,(\mathbb{S}^n(K))$ is an injection.

We close this section with

Conjecture 1.13: *For any $n \geq 1$, the restriction map*

$$\rho_n(K^{n+1}, \mathbb{S}^n(K)) : Aut\,(K^{n+1}, \mathbb{S}^n(K)) \to Aut\,(\mathbb{S}^n(K))$$

leads to:

(a) *a surjection, provided K is an infinite field with characteristic different from two;*

(b) *an isomorphism, provided K is a real field. Consequently, $Aut\,(\mathbb{S}^n(K)) = O(n+1, K)$.*

2. Polynomial automorphisms for higher dimensions

Now, we aim to present another set of generators of the group

$$\mathrm{Aut}(K[X_0, X_1, X_2]/(X_0X_1 + X_2^2 - 1)).$$

Consider the set $\{\pm 1\}\times K^*\times K[X]$ with the multiplication

$$\circ : (\{\pm 1\}\times K^*\times K[X])\times(\{\pm 1\}\times K^*\times K[X]) \to \{\pm 1\}\times K^*\times K[X]$$

given by

$$(\varepsilon,\lambda,p(X))\circ(\varepsilon',\lambda',p'(X)) = (\varepsilon\varepsilon',\lambda\lambda',\varepsilon'{\lambda'}^{-1}p({\lambda'}^{-1}X)+p'(X))$$

for

$$(\varepsilon,\lambda,p(X)),(\varepsilon',\lambda',p'(X))\in\{\pm 1\}\times K^*\times K[X].$$

Then, the pair $(\{\pm 1\}\times K^*\times K[X],\circ)$ is a group, where $(1,1,0)$ is its unit element and $(\varepsilon,\lambda,p(X))^{-1}=(\varepsilon,\lambda^{-1},-\varepsilon\lambda p(\lambda X))$. Furthermore, for $(\varepsilon,\lambda,p(X))\in\{\pm 1\}\times K^*\times K[X]$, the map

$$\varphi' : K[X_0,X_1,X_2]\to K[X_0,X_1,X_2]$$

given by:

$$\begin{cases}\varphi'(X_0)=\lambda X_0,\\ \varphi'(X_1)=\lambda^{-1}X_1-2\varepsilon X_2p(\lambda X_0)-\lambda X_0p^2(\lambda X_0),\\ \varphi'(X_2)=\varepsilon X_2+\lambda X_0p(\lambda X_0)\end{cases}$$

is a K-algebra automorphism with the inverse given by:

$$\begin{cases}{\varphi'}^{-1}(X_0)=\lambda^{-1}X_0,\\ {\varphi'}^{-1}(X_1)=\lambda(X_1+2X_2p(X_0)-X_0p^2(X_0)),\\ {\varphi'}^{-1}(X_2)=\varepsilon(X_2-X_0p(X_0)).\end{cases}$$

We recall that a K-automorphism of $K[X_0,\dots,X_n]$ is called **elementary** it it has a form

$$(X_0,\dots,X_n)\mapsto(X_0,\dots,X_{i-1},\lambda X_i+p,X_{i+1},\dots,X_n],$$

where $\lambda\in K^*$ and $p\in K[X_0,\dots,X_{i-1},X_{i+1},\dots,X_n]$ for some $i=0,\dots,n$. A K-automorphism of $K[X_0,\dots,X_n]$ is **tame** if it is a composition of elementary automorphisms. Note that φ' is a tame automorphism of $K[X_0,X_1,X_2]$ as the composition of three elementary automorphisms.

Furthermore, the relation $\varphi'(X_0)\varphi'(X_1)+\varphi'(X_2)^2 = X_0X_1+X_2^2$ shows that φ' yields a K-automorphism $\varphi : K[X_0,X_1,X_2]/(X_0X_1+X_2^2-1)\to K[X_0,X_1,X_2]/(X_0X_1+X_2^2-1)$. It is easy to check that the assignment above determines a group monomorphism

$$\Phi : \{\pm 1\}\times K^*\times K[X]\longrightarrow \mathrm{Aut}(K[X_0,X_1,X_2]/(X_0X_1+X_2^2-1)).$$

If $p(X) = a_nX^n + \cdots + a_1X + a_0$ with $a_n \neq 0$ and $(\varepsilon, \lambda, p(x)) \in \{\pm 1\} \times K^* \times K[X]$, then clearly

$$(\varepsilon, \lambda, p(X)) = (\varepsilon, \lambda, 0) \circ (1, 1, p(X)) = (1, \lambda, 0) \circ (\varepsilon, 1, 0) \circ (1, 1, p(X)) =$$
$$(1, \lambda, 0) \circ (\varepsilon, 1, 0) \circ (1, 1, a_0) \circ \cdots \circ (1, 1, a_nX^n).$$

This shows that the image of the monomorphism

$$\Phi : \{\pm 1\} \times K^* \times K[X] \longrightarrow \mathrm{Aut}(K[X_0, X_1, X_2]/(X_0X_1 + X_2^2 - 1))$$

is generated by the following three types of automorphisms:

(1) (hyperbolic rotations) $H_\lambda(X_0) = \lambda X_0$, $H_\lambda(X_1) = \lambda^{-1}X_1$, $H_\lambda(X_2) = X_2$ for $\lambda \in K^*$;

(2) (the symmetry automorphism) $S(X_0) = X_0$, $S(X_1) = X_1$, $S(X_2) = -X_2$ and

(3) (triangular automorphisms)

$$\Delta_{\lambda,n}(X_0) = X_0, \quad \Delta_{\lambda,n}(X_1) = X_1 - 2\lambda X_2X_0^n - \lambda^2X_0^{2n+1},$$
$$\Delta_{\lambda,n}(X_2) = X_2 + \lambda X_0^{n+1}$$

for $\lambda \in K^*$ and $n \geq 0$.

But $(1, \lambda, 0) \circ (1, 1, X^n) \circ (1, \lambda^{-1}, 0) = (1, 1, \lambda^{n+1}X^n)$ and the polynomial $X^{n+1} - \lambda$ has a root in K because the field K is algebraically closed. Finally, in the light of Corollary 1, we can state for an algebraically closed field K

Remark 2.1: The group $\mathrm{Aut}(K[X_0, X_1, X_2]/(X_0X_1 + X_2^2 - 1))$ is generated by the automorphisms (1)–(3) from Corollary 1 and

$$(4') \qquad \text{triangular } \Delta_{1,n}(X_0) = X_0, \Delta_{1,n}(X_1) = X_1 - 2X_2X_0^n - X_0^{2n+1},$$
$$\Delta_{1,n}(X_2) = X_2 + X_0^{n+1} \quad \text{for } n \geq 0.$$

In particular, the subgroup $\mathrm{Aut}_1(K[X_0, X_1, X_2]/(X_0X_1 + X_2^2 - 1))$ of automorphisms with degree at most one is generated by: H_λ, I, S and $\Delta_{1,0}$.

Hence, generators of the group $\mathrm{Aut}\,(K[X_0, X_1, X_2]/(X_0X_1 - 1 + X_2^2))$ described in Remark 2.1 correspond to the following ones of

$$\mathrm{Aut}\,(K[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2 - 1))$$

in matrix forms:

Any **hyperbolic rotation** H_λ yields the rotation

$$H'_\lambda : K[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2 - 1) \to K[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2 - 1)$$

given by the matrix $\begin{pmatrix} \frac{\lambda^{-1}+\lambda}{2} & \frac{i(\lambda^{-1}-\lambda)}{2} & 0 \\ \frac{-i(\lambda^{-1}-\lambda)}{2} & \frac{\lambda^{-1}+\lambda}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from the special orthogonal group $SO(3,K)$ for $\lambda \in K^*$.

The **involution map** I is sent to the map

$$I' : K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1) \to K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1)$$

given also by the orthogonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The **triangular map** $\Delta_{1,n}$ produces a map

$$\Delta'_{1,n} : K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1) \to K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1)$$

given by:

$$\Delta'_{1,n}(X_0) = (1 - \frac{(X_0+iX_1)^{2n}}{2})X_0 - i\frac{(X_0+iX_1)^{2n}}{2}X_1 - (X_0+iX_1)^n X_2,$$

$$\Delta'_{1,n}(X_1) = -i\frac{(X_0+iX_1)^{2n}}{2}X_0 + (1 + \frac{(X_0+iX_1)^{2n}}{2})X_1 - i(X_0+iX_1)^n X_2$$

and

$$\Delta'_{1,n}(X_2) = (X_0+iX_1)^n X_0 + i(X_0+iX_1)^n X_1 + X_2.$$

Observe that $\Delta'_{1,n}$ can be also given by the matrix from $SO(3,K[X_0+iX_1])$:

$$\begin{pmatrix} 1 - \frac{(X_0+iX_1)^{2n}}{2} & -i\frac{(X_0+iX_1)^{2n}}{2} & -(X_0+iX_1)^n \\ -i\frac{(X_0+iX_1)^{2n}}{2} & 1 + \frac{(X_0+iX_1)^{2n}}{2} & -i(X_0+iX_1)^n \\ (X_0+iX_1)^n & i(X_0+iX_1)^n & 1 \end{pmatrix}.$$

The **symmetry map** S is sent to

$$S' : K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1) \to K[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1)$$

given also by the orthogonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Given $\varphi \in \mathrm{Aut}(K^{n+1}, \mathbb{S}^n(K))$, let $\varphi^{-1} : K^{n+1} \to K^{n+1}$ be such that $\varphi \circ \varphi^{-1} = \varphi^{-1} \circ \varphi = \mathrm{id}_{K^{n+1}}$. Then, $(J\varphi^{-1})(\varphi(x))(J\varphi)(x) = I_{n+1}$ and consequently $((J\varphi)(x))^{-1} = (J\varphi^{-1})(\varphi(x))$. Since $\varphi \in \mathrm{Aut}\,(K^{n+1}, (\mathbb{S}^n))$, by Propositions 1.3 and 1.5, we have $\det(J\varphi)(x) = \det(J\varphi)(0) = \pm 1$. Furthermore, the relation $(J\varphi)(x)^t\varphi(x) = x$ yields $\varphi(x) = (((J\varphi)(x))^{-1})^t x$ for $x \in K^{n+1}$.

Therefore, by composing with $(J\varphi)(0)^{-1}$ we can always assume that φ is such that $(J\varphi)(0) = I_{n+1}$ and $\det(J\varphi)(x) = 1$ for $x \in K^{n+1}$. Hence, given $\varphi \in \mathrm{Aut}\,(K^{n+1}, \mathbb{S}^n)$, we can always assume, by composing with an element of $O(n+1,K)$, that $\varphi(x) = \tilde{\varphi}(x)x$ for $x \in K^{n+1}$, $\tilde{\varphi} \in SL(n+1, K[X_0,\ldots,X_n])$

with $\tilde{\varphi}(0) = I_{n+1}$ for the special linear group $SL(n+1, K[X_0, \ldots, X_n])$ over the polynomial ring $K[X_0, \ldots, X_n]$.

Certainly, $O(n+1, K) \subseteq \operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K)), \operatorname{Aut}(\mathbb{S}^n(K))$ for $n \geq 1$ and any field K. Now, we plan to find other groups included into $\operatorname{Aut}(K^{n+1}, \mathbb{S}^n(K))$.

Let K be a commutative ring and denote by $S_{m,n}(K)$ the set of all couples (a, λ), where $a \in M_{m,n}(K)$ is an $m \times n$-matrix over K and $\lambda \in M_m(K)$ is an $m \times m$-matrix over K such that

$$\lambda + \lambda^t + aa^t = 0,$$

where a^t denotes the transpose of the matrix a.

We also consider the case $n = 0$ and $S_{m,0}(K)$ consists of the additive group of skewsymmetric $m \times m$ matrices over K.

For $m, n \geq 1$, we define a group structure on $S_{m,n}(K)$ by

$$(a, \lambda).(a', \lambda') = (a + a', \lambda + \lambda' - aa'^t).$$

The definition is clearly correct, the unit element is $(0, 0)$, the inverse of (a, λ) is $(-a, \lambda^t)$ and associativity is easily checked.

Observe that for $1/2 \in K$, there is an obvious bijection $K^{\frac{m(m-1)}{2}+mn} \xrightarrow{\cong} S_{m,n}(K)$.

The following proposition is easy to check.

Proposition 2.2: *For $i \in K$ with $i^2 = -1$, there is a group monomorphism*

$$\varphi_{m,n}(K) : S_{m,n}(K) \longrightarrow SO(2m+n, K)$$

given by

$$\varphi_{m,n}(K)(a, \lambda) = \begin{pmatrix} I_m + \lambda & i\lambda & a \\ i\lambda & I_m - \lambda & ia \\ -a^t & -ia^t & I_n \end{pmatrix},$$

for $n \geq 1$ and

$$\varphi_{m,0}(K)(\lambda) = \begin{pmatrix} I_m + \lambda & i\lambda \\ i\lambda & I_m - \lambda \end{pmatrix},$$

for $n = 0$, where I_m is the unit $m \times m$-matrix.

For $(a, \lambda) \in S_{m,n}(K[X_1, \ldots, X_m])$ define a polynomial map

$$\Phi_{m,n}(K)(a, \lambda) : K^{2m+n} \longrightarrow K^{2m+n}$$

by

$$\Phi_{m,n}(K)(a,\lambda)(x,y,z)=\begin{pmatrix} I_m+\lambda(x+iy) & i\lambda(x+iy) & a(x+iy) \\ i\lambda(x+iy) & I_m-\lambda(x+iy) & ia(x+iy) \\ -a(x+iy)^t & -ia(x+iy)^t & I_n \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)$ and $z=(z_1,\dots,z_n)$.

The next proposition is again an easy checking.

Proposition 2.3: *The above map $\Phi_{m,n}(K)(a,\lambda)$ is a polynomial automorphism of K^{2m+n}, $\Phi_{m,n}(K)(a,\lambda)\in G_{2m+n}(K)$ for $(a,\lambda)\in S_{m,n}(K[X_1,\dots,X_m])$ and*

$$\Phi_{m,n}(K): S_{m,n}(R[X_1,\dots,X_m]) \longrightarrow \mathrm{Aut}(K^{2m+n},\mathbb{S}^{2m+n-1}(K))$$

is a group monomorphism.

In this way, we get that the group monomorphism

$$\Phi_{m,n}(K): S_{m,n}(R[X_1,\dots,X_m]) \longrightarrow \mathrm{Aut}(K^{2m+n},\mathbb{S}^{2m+n-1}(K))$$

leads to an imbedding

$$K[X_1\dots,X_m]^{\frac{m(m-1)}{2}+mn} \longrightarrow \mathrm{Aut}\,(K^{2m+n},\mathbb{S}^{2m+n-1}(K)).$$

provided $1/2\in K$.

Observe, in particular that $\mathrm{Aut}(K^{2n},\mathbb{S}^{2n-1}(K))$ contains the subgroups

$$\Phi_{j,2k}(K)(S_{j,2k}(K[X_1,\dots,X_j]))$$

for all j,k such that $j+k=n$ and $\mathrm{Aut}(K^{2n+1},\mathbb{S}^{2n}(K))$ contains the subgroups

$$\Phi_{j,2k+1}(K)(S_{j,2k+1}(K[X_1,\dots,X_j]))$$

for all j,k such that $j+k=n$.

At the end, let K be a field of characteristic different from two and with $i\in K$. Then, we can easily deduce from [6] that the group $\mathrm{Aut}\,(\mathbb{S}^2(K))$ is generated by $O(3,K)$ and the image of $\Phi_{1,1}(K)$. We close this section with

Conjecture 2.4: *For any $n\geq 1$:*

(a) the group $O(2n+1,K)$ and the images of $\Phi_{j,2k+1}(K)$ with $j+k=n$ generate $Aut(K^{2n+1},\mathbb{S}^{2n}(K))$;

(b) the group $O(2n,K)$ and the images of $\Phi_{j,2k}(K)$ with $j+k=n$ generate $Aut(K^{2n},\mathbb{S}^{2n-1}(K))$.

We point that that Conjecture 2.4 holds for $n = 2$.

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